E_∞ CELL MODELS FOR FREE AND BASED LOOP SPACE COHOMOLOGY

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ABSTRACT. We construct E_∞ cell algebra models for the free and based loop spaces on a simply-connected topological space. Techniques from rational homotopy theory are exploited throughout.

1. Introduction

The purpose of this paper is to use methods from rational homotopy theory to construct E_∞ *cell models* for the normalised singular cochain algebras of the free and based loop spaces on a simply-connected topological space. The normalised singular cochains with coefficients in a commutative ring R, denoted $N^*(-)$, is a functor from spaces to E_∞ algebras over R (see, for example, [2]). The E_∞ algebras over R form a closed model category in which the fibrations are the surjections and the weak equivalences are the quasi-isomorphisms. The cofibrations are retracts of *cell extensions* (see Section 2), which are the E_∞ analogues of the KS extensions, or relative Sullivan algebras, of rational homotopy theory. A *cell model* for a morphism $\varphi : A \to B$ is a factorisation of φ as a cell extension followed by a weak equivalence. A *cell model* for A is a cell model of the unit map. Abusing notation, we refer to the weak equivalence itself as the model. The work of Mandell [11] shows that if R is the algebraic closure of $\mathbf{F}_p$, then the cell model of the cochain algebra of a space captures the p-adic homotopy type of the space. In particular, the cell model carries all of the information on the Steenrod operations in cohomology. This fact, along with the fact that every morphism has a cell model, addresses the difficulties encountered when trying to use commutative algebras to model p-local maps and spaces.

Let LX and ΩX be the free and based loop spaces, respectively, on the pointed, simply-connected topological space X. We prove the following Main Theorem and Important Corollary.

Main Theorem. *Let $m_X : (\mathbf{E}(V), d) \xrightarrow{\sim} N^*(X)$ be a cell algebra model. Then there exists a cell algebra model $(\mathbf{E}(V \oplus sV), d) \xrightarrow{\sim} N^*(LX)$, and the cell extension $(\mathbf{E}(V), d) \to (\mathbf{E}(V \oplus sV), d)$ models the evaluation map $ev : LX \to X$.*

Important Corollary. *With the notation of the Main Theorem, there exists a cell model of the form $(\mathbf{E}(sV), d) \xrightarrow{\sim} N^*(\Omega X)$, and the quotient map $(\mathbf{E}(V \oplus sV), d) \to (\mathbf{E}(sV), d)$ models the inclusion $\Omega X \subset LX$.*

In fact, the Main Theorem and the Important Corollary hold for any cofibrant models. They are stated in full generality as Theorem 4.1 and Corollary 4.6, respectively. Similar results have been found by Chataur and Thomas [4] using an operadic Hochschild complex, and by Fresse [6] using a derived functor of the left

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adjoint to a mapping functor. The nice thing about the Important Corollary, as compared to the usual situation when working p-locally (see [1, 9, 14]), is that the construction may be iterated:

Corollary 1.1. *If X is q-connected, then there exists a cell model of the form $(\mathbf{E}(s^qV), d) \xrightarrow{\sim} N^*(\Omega^q X)$.*

Example 1.2. A cell algebra model $\mathbf{E}(V) \xrightarrow{\sim} N^*(S^{2n+1})$ determines cell algebra models $\mathbf{E}(sV) \xrightarrow{\sim} N^*(\Omega S^{2n+1})$ and $\mathbf{E}(s^2V) \xrightarrow{\sim} N^*(\Omega^2 S^{2n+1})$. So all of the Steenrod operations in $H^*(\Omega^2 S^{2n+1}; \mathbf{F}_p)$ are lurking somewhere in $\mathbf{E}(V)$.

The outline of the paper is as follows. Section 2 covers the definitions and basic properties of operads, algebras, and cell algebras, and our E_∞ operad of choice, the Barratt-Eccles operad. In Section 3 we construct explicit cylinder and cone objects for cell $\mathcal{O}$-algebras for certain operads $\mathcal{O}$ including the Barratt-Eccles operad. In Section 4 we assemble various facts from [11] to show that the pushout of cell models gives a cell model of the pullback of spaces (Lemma 4.2) and use this result to prove the Main Theorem and hence deduce the Important Corollary. We end with some computational examples in Section 5.

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2. Operads and their algebras

In this section we fix notation and recall some definitions. Throughout the paper wee work over a commutative ground ring R.

Differential graded (DG) R-modules are $\mathbf{Z}$-graded, and we use the convention $M^j = M_{-j}$. Define the R-free chain complex I by $I_0 = R\{e', e''\}$, $I_1 = R\{s\}$, $ds = e' - e''$. The augmentation $I \to R$, sending e' and e'' to 1, is a quasi-isomorphism. The *cylinder* on a DG R-module M is $IM := I \otimes M$. Note that $IM = M' \oplus M'' \oplus sM$, where $M' = e' \otimes M \cong M$, $M'' = e'' \otimes M \cong M$, and $sM = s \otimes M$. Since I is a semifree R-module, the augmentation on I induces a quasi-isomorphism $IM \xrightarrow{\sim} M$. The *cone* on M is the quotient $CM := IM/(e' - e'') = M \oplus sM$, and we have a quasi-isomorphism $CM \xrightarrow{\sim} 0$. The *suspension* on M is the quotient $sM := CM/M$. Suspension is an isomorphism of lower degree +1: $s : M_j \cong (sM)_{j+1}$.

An *operad* $\mathcal{O}$ consists of DG R-modules $\mathcal{O}(n)$, $n \geq 0$, together with a unit map $R \to \mathcal{O}(1)$, a right action of the symmetric group Σ_n on each $\mathcal{O}(n)$, $n \geq 1$, and chain maps
$$\mathcal{O}(n) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_n) \to \mathcal{O}(j_1 + \cdots j_n)$$
for $n \geq 1$, $j_s \geq 0$, called the *composition products*, that are required to be associative, unital, and equivariant. See Kříž and May [10] for the details.

An $\mathcal{O}$-*algebra* is a DG R-module A together with chain maps
$$\mathcal{O}(n) \otimes A^{\otimes n} \to A$$
for $n \geq 0$ that are associative, unital, and equivariant. Once again, see Kříž and May [10]. Coproducts other than direct sums and tensor products shall be denoted by the symbol $\vee$.

The *free* $\mathcal{O}$-*algebra* on a DG R-module (V, d) is defined by
$$\mathbf{O}(V, d) = \bigoplus_{n \geq 0} \left(\mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n} \right).$$

We will abuse notation and write $(\mathbf{O}(V), d)$ for an $\mathcal{O}$-algebra that is free if differentials are ignored.

For a simplicial set Y, denote by $N_*(Y)$ and $N^*(Y)$ the normalised chains and cochains on Y, respectively. If X is a space we take $N_*(X) = N_*(S(X))$ where $S(X)$ is the singular simplicial set on X.

Let X be a set. Denote by $W(X)$ the *standard simplicial resolution of* X, where $W(X)_n = X^{n+1}$, $n \geq 0$. Face and degeneracy maps are defined by deletion and repetition, respectively.

The *Barratt-Eccles operad* $\mathcal{E}$ is defined as $\mathcal{E}(n) = N_*(W(\Sigma_n))$, with composition products determined by block permutations. The reader is referred to Berger and Fresse [2] for a detailed description of the composition product. For $n \geq 0$, $\mathcal{E}(n)$ is an $R[\Sigma_n]$-free resolution of R, so $\mathcal{E}$ is an E_∞ operad in the sense of Kříž and May. In addition, Berger and Fresse showed that the category of $\mathcal{E}$-algebras has a particularly nice model structure. The fibrations are the surjections, the weak equivalences are the quasi-isomorphisms, and the cofibrations are retracts of cell extensions, which we now define.

Let $\mathcal{O}$ be an operad. An $\mathcal{O}$-algebra morphism $j : A \to B$ is called a *cell extension* (*relative cell inclusion* in the language of Mandell [11]) if

(1) forgetting differentials, $B \cong A \vee \mathbf{O}(V)$,
(2) j is the canonical inclusion, and
(3) V is the union of a nested sequence of submodules $V(k)$, $k \geq 0$, such that $V(0)$ and $V(k)/V(k-1)$, $k \geq 0$, are R-free, $dV(0) \subset A$, and $d(V(k)) \subset A \vee \mathbf{O}(V(k-1))$ for $k \geq 1$.

In particular, we may write $V(k) = V_k \oplus V(k-1)$ with $V_k \cong V(k)/V(k-1)$ and $d(V_k) \subset A \vee \mathbf{O}(V(k-1))$. A *cell algebra* is a cell extension of R. A *cell model* of a morphism $\varphi : A \to A'$ is a factorisation of φ as a cell extension followed by a weak equivalence. A cell model of an $\mathcal{O}$-algebra A is a cell model of the unit morphism $R \to A$.

Let A be an element of a closed model category, and fix a cylinder object IA. Set $LA = A \vee_{A\vee A} IA$ and $SA = R \vee_A LA$, where R is the terminal object in the category. Both LA and SA depend upon choice of cylinder object, but only up to weak equivalence.

3. Cylinder objects and acyclic closures

In this section we construct an explicit cylinder object and acyclic closure for a given cell algebra. Let $\mathcal{O}$ be an operad such that the $\mathcal{O}$-algebras form a closed model category where the fibrations are the surjections and the weak equivalences are the quasi-isomorphisms. This condition is satisfied, for example, by the associative algebra operad, the commutative algebra operad if $R \supseteq \mathbf{Q}$, the Barratt-Eccles operad, or any cofibrant operad. Essentially, we want the free $\mathcal{O}$-algebra functor to preserve quasi-isomorphisms.

Proposition 3.1. *Let $A = (\mathbf{O}(V), d)$ be a cell $\mathcal{O}$-algebra. Set $V' = V'' = V$. Then the fold map $\nabla : A \vee A \to A$ has a surjective cell model*

$$A \vee A \rightarrowtail IA \overset{\sim}{\twoheadrightarrow} A$$

where $IA = (\mathbf{O}(V' \oplus V'' \oplus sV), d)$. Furthermore, if $x \in V_k$, then

$$d(sx) - x' + x'' \in \mathbf{O}(V'(k-1) \oplus V''(k-1) \oplus sV(k-1)).$$

Proof. We construct IA recursively, proceeding along the filtration on V. First we introduce some notation. For $k \geq 0$, let $V'(k)$ and $V''(k)$ be isomorphic copies of $V(k)$. The images of $x \in V(k)$ in $V'(k)$ and $V''(k)$ will be denoted x' and x'', respectively. Set $A(k) = (\mathbf{O}(V(k)), d)$ and $IA(k) = (\mathbf{O}(V'(k) \oplus V''(k) \oplus sV(k)), d)$.

Write $A'(k)$ and $A''(k)$ for the subalgebras of $IA(k)$ generated by $V'(k)$ and $V''(k)$, respectively.

Suppose that we have constructed the algebra $IA(k)$ such that the epimorphism $\eta_k : IA(k) \to A(k)$ defined by $\eta_k(x') = \eta_k(x'') = x$, $\eta_k(sx) = 0$ for all $x \in V(k)$, is a weak equivalence. Recall that $V(k+1) = V_{k+1} \oplus V(k)$, with $d(V_{k+1}) \subset A(k)$. Let x be a basis element of V_{k+1}. By definition, $dx' \in A'(k)$, $dx'' \in A''(k)$, and $\eta_k(d(x' - x'')) = 0$. Since η_k is a trivial fibration, $\ker \eta_k$ is contractible, and so the cycle $d(x' - x'') = d\Phi$ for some $\Phi \in \ker \eta_k \subset IA(k)$. Extend the differential to $sV(k+1)$ by setting $dsx = x' - x'' - \Phi$. Set $\eta_{k+1}(v') = \eta_{k+1}(v'') = v$ and $\eta_{k+1}(sv) = 0$ to extend η_k to η_{k+1} .

We need to show that η_{k+1} is a weak equivalence. To this end, define filtrations on $IA(k+1)$ and $A(k+1)$ by $F_0 = IA(k+1)$, $F_1 = IA(k)$, and $G_0 = A(k+1)$, $G_1 = A(k)$. The morphism η_{k+1} preserves the filtrations and so defines a morphism of strongly convergent right-half-plane spectral sequences. At the E_0-term, the induced morphism is

$$E_0(\eta_{k+1}) = \mathbf{O}(\epsilon) \vee \eta_k : \mathbf{O}(IV_k) \vee IA(k) \to \mathbf{O}(V_k) \vee A(k)$$

where IV_k is the cylinder on $(V_k, 0)$. Since $IV_k \xrightarrow{\sim} V_k$ and $\mathbf{O}(-)$ preserves weak equivalences, $E_0(\eta_{k+1})$ is a weak equivalence. □

The *acyclic closure*, or *cone*, of A is the algebra $CA := R \vee_A IA$.

Corollary 3.2. *$A \to CA$ is a cell extension, and the augmentation $A \to R$ factors as $A \rightarrowtail CA \overset{\sim}{\twoheadrightarrow} R$.*

Proof. Apply $R \vee_A -$ to $A \vee A \rightarrowtail IA \overset{\sim}{\twoheadrightarrow} A$. □

4. Models for free and based loop spaces

In this section we specialise to an E_∞-operad $\mathcal{E}$, such as the Barratt-Eccles operad. We assemble facts from Mandell [11] to prove Lemma 4.2. We then use a characterisation of LX as a pullback to prove Theorem 4.1 and deduce Corollary 4.6.

Let MX be the space of free paths on the simply-connected space X. We can describe LX as the pullback of the diagram

$$\begin{tikzcd} LX \arrow[r] \arrow[d, "ev"] & MX \arrow[d, "{(e_0, e_1)}"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

where ev is evaluation at the basepoint of the loop and e_0, e_1 are evaluations at the endpoints of the path.

Theorem 4.1. *Let $m_X : A \xrightarrow{\sim} N^*(X)$ be a cofibrant model. Then the composite*

$$A \xrightarrow{\sim} N^*(X) \xrightarrow{ev^*} N^*(LX)$$

has a cofibrant model

$$A \rightarrowtail LA \overset{\sim}{\twoheadrightarrow} N^*(LX)$$

where $LA := A \vee_{A \vee A} IA$. If $A = (\mathbf{E}(V), d)$ is a cell algebra, then $LA = (\mathbf{E}(V \oplus sV), d)$ and $A \rightarrowtail LA$ is a cell extension.

As mentioned above, the following lemma is implicit in Mandell [11].

Lemma 4.2. *Let $\pi : E \to B$ be a fibration, and let $f : X \to B$ be continuous. Suppose there exists a commutative diagram*

$$\begin{tikzcd}
A_X \arrow[d, "\theta_X"', "\sim"] & A_B \arrow[l, "\theta_f"'] \arrow[r, tail] \arrow[d, "\theta_B"', "\sim"] & A_E \arrow[d, "\theta_E"', "\sim"] \\
N^*(X) & N^*(B) \arrow[l] \arrow[r] & N^*(E)
\end{tikzcd}$$

of E_∞ algebras and morphisms, in which the algebras in the top row are cofibrant. Then the map induced by pushout

$$\theta : A_X \vee_{A_B} A_E \to N^*(X \times_B E)$$

is a weak equivalence.

Proof. First we suppose that θ_f is a cofibration, and that all vertical morphisms are fibrations. Following Mandell [11], let $N(\beta(A_X, A_B, A_E))$ denote the normalised chains on the simplicial bar construction. The composition of natural maps

$$N(\beta(A_X, A_B, A_E)) \to A_X \vee_{A_B} A_E \xrightarrow{\theta} N^*(X \times_B E)$$

is a weak equivalence by [11, Lemma 5.2], while the first map is a weak equivalence by [11, Theorem 3.5]. By the two-out-of-three rule, θ is a weak equivalence.

If the vertical arrows are not necessarily fibrations, use the closed model category structure on E_∞ algebras to form the diagram

$$\begin{tikzcd}
A_X \arrow[dd, "\theta_X"', "\sim"] & & A_B \arrow[ll, "\theta_f"'] \arrow[r, tail] \arrow[d, "\psi_B"', "\sim"] & A_E \arrow[d, "\psi_E"', "\sim"] \\
& B_X \arrow[dl, "\varphi_X"', "\sim"] & B_B \arrow[l, tail, "\varphi_f"'] \arrow[r, tail, "\varphi_\pi"] \arrow[d, two heads, "\varphi_B"', "\sim"] & B_E \arrow[d, two heads, "\varphi_E"', "\sim"] \\
N^*(X) & & N^*(B) \arrow[ll, "f^*"'] \arrow[r, "pi^*"] & N^*(E)
\end{tikzcd}$$

in which $\varphi_B \circ \psi_B$, $\varphi_X \circ \varphi_f$ and $\varphi_E \circ \varphi_\pi$ are factorisations of θ_B, $f^* \circ \varphi_B$ and $\pi^* \circ \varphi_B$, respectively, into a cofibration followed by a trivial fibration, and ψ_E is a lift for the diagram

$$\begin{tikzcd}
A_B \arrow[r] \arrow[d, tail] & B_E \arrow[d, two heads, "\sim"] \\
A_E \arrow[r] & N^*(E).
\end{tikzcd}$$

By the first paragraph of the proof, the pushout morphism $\varphi : B_X \vee_{B_B} B_E \to N^*(X \times_B E)$ is a weak equivalence.

If θ_X is a surjection, then factor $\varphi_f \circ \psi_B$ as $\xi_X \circ \psi_f$, where $\psi_f : A_B \rightarrowtail C_X$ is a cofibration and $\xi_X : C_X \overset{\sim}{\twoheadrightarrow} B_X$ is a trivial fibration. The pushout morphism $\tilde{\psi} : C_X \vee_{A_B} A_E \to B_X \vee_{B_B} B_E$ induced by ξ_X, ψ_B, and ψ_E, is a weak equivalence by [11, Theorem 3.2]. Now lift $\varphi_X \circ \xi_X$ through θ_X to define a weak equivalence $\eta_X : C_X \xrightarrow{\sim} A_X$ such that $\theta_f = \eta_X \circ \psi_f$. The pushout morphism $\tilde{\eta} : C_X \vee_{A_B} A_E \to A_X \vee_{A_B} A_E$ induced by η_X is a weak equivalence by [11, Theorem 3.2]. By uniqueness of pushout, $\theta \circ \tilde{\eta} = \varphi \circ \tilde{\psi}$. It follows that θ is a weak equivalence.

If θ_X is not necessarily surjective, then factor it as $\theta_X = p_X \circ i_X$, where $p_X : U_X \to A_X$ is a trivial fibration and i_X is a cofibration. By the two-out-of-three rule, i_X is a weak equivalence. By the previous paragraph, p_X, θ_B, and θ_E induce a weak equivalence $\tilde{\theta} : U_X \vee_{A_B} A_E \to N^*(X \times_B E)$. Furthermore, the map of pushouts

$i : A_X \vee_{A_B} A_E \to U_X \vee_{A_B} A_E$ induced by i_X and the identities on A_B and A_E is a weak equivalence by [11, Theorem 3.2]. Thus $\theta = \tilde{\theta} \circ i$ is a weak equivalence. □

Setting $X = *$ we obtain the E_∞ analogue of the theorem on the model of the fibre [7, 8, 5, 13, 12].

Corollary 4.3. *(cf. Chataur [3]) Let $\pi : E \to B$ be a fibration with fibre F. Then $R \vee_{A_B} A_E$ is a model for $N^*(F)$.*

Of course, Corollary 4.3 applies to the homotopy fibre of an arbitrary map.

Lemma 4.4. *Let $m_X : A \xrightarrow{\sim} N^*(X)$ be a cofibrant model. Then the fold map $A \vee A \xrightarrow{\nabla} A$ models Δ^*. That is, there exists a weak equivalence $m_{X\times X} : A \vee A \xrightarrow{\sim} N^*(X \times X)$ such that the diagram*

$$\begin{tikzcd}
A \vee A \arrow[r, "\nabla"] \arrow[d, "\sim"] & A \arrow[d, "\sim"] \\
N^*(X \times X) \arrow[r, "\Delta^*"] & N^*(X)
\end{tikzcd}$$

commutes.

Proof. Denote by $\pi_1, \pi_2 : X \times X \to X$ the canonical projections.

The product $X \times X$ is the pullback of the diagram $X \to * \leftarrow X$. Since $N^*(*) = R$ and m_X commutes with unit morphisms, Lemma 4.2 states that the pushout morphism $m_{X\times X} : A_X \vee A_X \xrightarrow{\sim} N^*(X \times X)$ is a weak equivalence. Since $\Delta \circ \pi_i = 1_X$ for $i = 1, 2$, the compositions $m_X \circ \nabla$ and $\Delta^* \circ m_{X\times X}$ both make the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A_X \arrow[r, two heads, "\sim"] \arrow[d] & N^*(X) \arrow[d, "\pi_1^*"] \\
A_X \arrow[r] \arrow[d, "\sim"] & A_X \vee A_X \arrow[dr, dotted] & N^*(X \times X) \arrow[d, "\Delta^*"] \\
N^*(X) \arrow[r, "\pi_2^*"] & N^*(X \times X) \arrow[r, "\Delta^*"] & N^*(X)
\end{tikzcd}$$

commute. Therefore $m_X \circ \nabla = \Delta^* \circ m_{X\times X}$. □

If A is an E_∞ algebra, denote by IA a cylinder object on A, and $j_1, j_2 : A \to IA$ the canonical inclusions.

Lemma 4.5. *Let $e = (e_0, e_1) : MX \to X \times X$. Given the hypotheses of Lemma 4.4, the canonical morphism $A \vee A \xrightarrow{j_1+j_2} IA$ is a cofibrant model for e^*. That is, there exists a weak equivalence $m_{MX} : IA \xrightarrow{\sim} N^*(MX)$ such that the diagram*

$$\begin{tikzcd}
A \vee A \arrow[r, "j_1+j_2"] \arrow[d, "m_{X\times X}"', "\sim"] & IA \arrow[d, "m_{MX}"', "\sim"] \\
N^*(X \times X) \arrow[r, "e^*"] & N^*(MX)
\end{tikzcd}$$

commutes. If A is a cell algebra, then IA may be chosen to be the cylinder of Proposition 3.1.

Proof. To construct m_{MX}, we note that $e_0 \simeq e_1 : MX \to X$.

Therefore $e_0^* \circ m_X \simeq e_1^* \circ m_X$ as E_∞ morphisms, and so the composite

$$A \vee A \xrightarrow{m_X \vee m_X} N^*(X) \vee N^*(X) \xrightarrow{e_0^*+e_1^*} N^*(MX)$$

factors as $A \vee A \xrightarrow{j_1+j_2} IA \xrightarrow{m_{MX}} N^*(MX)$.

Since e_0 is a homotopy equivalence, $e_0^* : N^*(X) \to N^*(MX)$ is a weak equivalence. Then m_{MX} is a weak equivalence because $m_{MX} \circ j_1 = e_0^* \circ m_X$.

Now, $e_0^* + e_1^* = e^* \circ (\pi_1^* + \pi_2^*)$ because $e_i = \pi_{i+1} \circ (e_0, e_1)$ for $i = 0, 1$, and $(\pi_1^* + \pi_2^*) \circ (m_X \vee m_X) = m_{X\times X}$ by uniqueness of pushout. We deduce that $e^* \circ m_{X\times X} = (e_0^* + e_1^*) \circ (m_X \vee m_X)$.

The above proof works for any cylinder object on A, in particular the cylinder of Proposition 3.1 if A happens to be a cell algebra. □

Now it is time for the

Proof of Theorem 4.1. Consider the diagram of continuous maps

$$X \xrightarrow{\Delta} X \times X \xleftarrow{e} MX \tag{1}$$

where $e = (e_0, e_1)$ and MX is the free path space on X. Recall that the pullback of (1) is the free loop space LX. Use Lemmas 4.4 and 4.5 to form the diagram

$$\begin{tikzcd}
A \arrow[d, "m_X", "\sim"'] & A \vee A \arrow[l, "\nabla"'] \arrow[r, tail] \arrow[d, "m_{X\times X}", "\sim"'] & IA \arrow[d, "m_{MX}", "\sim"'] \\
N^*(X) & N^*(X \times X) \arrow[l, "\Delta^*"'] \arrow[r, "e^*"] & N^*(MX).
\end{tikzcd}$$

By Lemma 4.2, the pushout morphism $m : A \vee_{A\vee A} IA \to N^*(LX)$ is a weak equivalence. Furthermore, if $j_A : A \rightarrowtail A \vee_{A\vee A} IA$ is the natural map, then $m \circ j_B = ev^* \circ m_X$ by construction.

The natural map $A \to LA$ is a cofibration since $A \vee A \rightarrowtail IA$ is one. If $A = (\mathbf{E}(V), d)$ is a cell algebra, with cellular filtration $V(k)$, then the filtration $(sV)(k) = s(V(k))$ exhibits $A \to LA$ as a cell extension, completing the proof. □

Corollary 4.6. *There exists a cell model of the form* $(\mathbf{E}(sV), d) \xrightarrow{\sim} N^*(\Omega X)$.

Proof of Corollary 4.6. The based loop space ΩX is the fibre of the evaluation map $ev : LX \to X$. By Corollary 4.3, the induced map $m_{\Omega X} : R \vee_A LA \to N^*(\Omega X)$ is a weak equivalence. If $A = \mathbf{E}(V)$ is a cell algebra, then we may assume $LA = \mathbf{E}(V \oplus sV)$, is a cell extension of A, whence $SA := R \vee_A LA = \mathbf{E}(sV)$ is a cell algebra. □

Remark 4.7. We may choose to characterise ΩX as the fibre of two other fibrations, namely, the evaluation maps $e : MX \to X \times X$ and $e_0 : PX \to X$, where PX is the space of paths ending at x_0. Of course, ev is the pullback of e along the diagonal $\Delta : X \to X \times X$ and e_0 is the pullback of e along $(id, x_0) : X \to X \times X$. The resulting models form the diagram of pushout squares

$$\begin{tikzcd}
A \arrow[d, tail] & A \vee A \arrow[l, tail] \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail] \\
CA & IA \arrow[l] \arrow[r] & LA
\end{tikzcd}$$

shows that $SA = R \vee_{A\vee A} IA = R \vee_A CA$. In practice, $R \vee_A CA$ is the easiest to compute.

5. Examples

In all the examples below we work over $\mathbf{F}_2$. Recall that $\mathcal{E}(2)_q = \mathbf{F}_2\pi \cdot e_q$, where π is the cyclic group of order 2 with generator τ, and $de_q = (1 + \tau)e_{q-1}$.

5.1. **Free and based loops on an Eilenberg-Mac Lane space.** We calculate the models for the free and based loop spaces on $K(\mathbf{Z}/2,n)$. We show that the model for the based loop space is the model for $K(\mathbf{Z}/2,n-1)$, and that the model for the free loop space splits as expected.

A model for $N^*(K(\mathbf{Z}/2,n);\mathbf{F}_2)$ is $A(n)=(\mathbf{E}(u,v),dv=u+e_n(u,u))$ (see [11]). Then $IA(n)=\mathbf{E}(u',v',u'',v'',su,sv)$, with $dsu=u'+u''$. Since

$$d(v'+v'')=d(su+e_n(u'',su)+e_n(su,u')+e_{n-1}(su,su)),$$

we may take

$$dsv=v'+v''+su+e_n(u'',su)+e_n(su,u')+e_{n-1}(su,su).$$

Setting $u'=u''$ and $v'=v''$, we find $LA(n)=\mathbf{E}(u,v,su,sv)$, with $dsu=0$ and $dsv=su+e_n(u,su)+e_n(su,u)+e_{n-1}(su,su)$. Then

$$SA(n)=(\mathbf{E}(su,sv),dsv=su+e_{n-1}(su,su))$$

is evidently $A(n-1)$, a model for $N^*(K(\mathbf{Z}/2,n-1);\mathbf{F}_2)$. The natural map $LA(n)\to SA(n)$ has a splitting, defined by $su\mapsto su$, $sv\mapsto sv+e_{n+1}(u,su)$, that defines an isomorphism $A(n)\vee SA(n)\xrightarrow{\cong}LA(n)$.

5.2. **The algebraic model of an elementary fibration.** Let X be a "nice" topological space, i.e. it is 2-complete and of finite 2-type. We want to completely determine the algebraic model of an elementary fibration over X. That is to say, we consider an algebraic model of the pullback square of the path fibration over $K(\mathbf{Z}/2,n)$ over a map $f:X\to K(\mathbf{Z}/2,n)$,

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & PK(\mathbf{Z}/2,n) \arrow[d] \\ X \arrow[r, "f"] & K(\mathbf{Z}/2,n) \end{tikzcd}$$

By Lemma 4.2, the model is given by the pushout square of algebras:

(2)
$$\begin{tikzcd} A_{X'} & CA(n) \arrow[l] \\ A_X \arrow[u] & A(n). \arrow[l] \arrow[u] \end{tikzcd}$$

where $A(n)$ is the model for $N^*(K(\mathbf{Z}/2,n);\mathbf{F}_2)$ given in Example 5.1, and $CA(n)$ is its acyclic closure (recall Corollary 3.2). The acyclic closure is a cell extension that models the path fibration

$$K(\mathbf{Z}/2,n-1)\longrightarrow PK(\mathbf{Z}/2,n)\longrightarrow K(\mathbf{Z}/2,n).$$

and its cofibre $SA(n)$ is $A(n-1)$, a model for $N^*(K(\mathbf{Z}/2,n-1);\mathbf{F}_2)$. The differential in $CA(n)$ extends that of $A(n)$, with $dsu=u$ and $dsv=v+su+e_{n-1}(su,su)+e_n(su,u)$.

We can now give a model of the map $X'\to X$. Let $\phi:A(n)\to A_X$ be a model for f. Set $z=\phi(u)$ and $z'=\phi(v)$. By the diagram (2), $A_{X'}=A_X\vee A(n-1)=A_X\vee\mathbf{E}(su,sv)$, and the differential in $A_{X'}$ satisfies $dsu=z$ and

$$dsv=u+e_{n-1}(su,su)+e_n(su,z)+z'.$$

5.3. **Loop spaces of some Postnikov 2-towers.** We are interested by determining a model of the loop space of the space X_1 obtained as the total space of the following elementary fibration:

$$\begin{array}{ccc} X_1 & \longrightarrow & PK(\mathbf{Z}/2, n+p) \\ \downarrow & & \downarrow \\ K(\mathbf{Z}/2, n) & \xrightarrow{f} & K(\mathbf{Z}/2, n+p) \end{array}$$

where f represents a Steenrod square Sq^p. Recall the algebra $A(k) = \mathbf{E}(u_k, v_{k-1})$ defined in Example 5.1. A model for f is given by

$$\phi : A(n+p) \to A(n)$$

where $\phi(u_{n+p}) = e_{n-p}(u_n, u_n)$ and $\phi(v_{n+p-1}) = \gamma_{n+p-1}$. Here γ_{n+p-1} is an element that satisfies $d\gamma_{n+p-1} = e_{n-p}(u_n, u_n) + e_{n+p}(e_{n-p}(u_n, u_n), e_{n-p}(u_n, u_n))$. Applying the formulas of the preceding examples we obtain

$$A_{X_1} = \mathbf{E}(u_n, v_{n-1}, w_{n+p-1}, t_{n+p-2})$$

together with the differential

$$\begin{aligned} du_n &= 0 \\ dv_{n-1} &= u_n + e_n(u_n, u_n) \\ dw_{n+p-1} &= e_{n-p}(u_n, u_n) \\ dt_{n+p-2} &= w_{n+p-1} + \gamma_{n+p-1} + e_{n+p-1}(w_{n+p-1}, w_{n+p-1}) \\ &\quad + e_{n+p}(e_{n-p}(u_n, u_n), w_{n+p-1}). \end{aligned}$$

A differential on the acyclic closure of A_{X_1},

$$CA_{X_1} = \mathbf{E}(u_n, v_{n-1}, w_{n+p-1}, t_{n+p-2}, u'_{n-1}, v'_{n-2}, w'_{n+p-2}, t'_{n+p-3})$$

is given by

$$\begin{aligned} du'_{n-1} &= u_n \\ dv'_{n-2} &= v_{n-1} + u'_{n-1} + e_{n-1}(u'_{n-1}, u'_{n-1}) + e_n(u_n, u'_{n-1}) \\ dw'_{n+p-2} &= w_{n+p-1} + e_{n-p}(u'_{n-1}, u_n) + e_{n-p-1}(u'_{n-1}, u'_{n-1}) \\ dt'_{n+p-3} &= t_{n+p-2} + w'_{n+p-2} + e_{n+p-2}(w'_{n+p-2}, w'_{n+p-2}) \\ &\quad + e_{n+p-1}(e_{n-p-1}(u'_{n-1}, u'_{n-1}), w'_{n+p-2}) + \gamma'_{n+p-2} \end{aligned}$$

such that in the cofibre $SA_{X_1} = \mathbf{E}(u'_{n-1}, v'_{n-2}, w'_{n+p-2}, t'_{n+p-3})$ we have:

$$d\gamma'_{n+p-2} = e_{n-p-1}(u'_{n-1}, u'_{n-1}) + e_{n+p-1}(e_{n-p-1}(u'_{n-1}, u'_{n-1}), e_{n-p-1}(u'_{n-1}, u_{n-1})).$$

Hence SA_{X_1} models $N^*(\Omega X_1; \mathbf{F}_2)$. The differential is

$$\begin{aligned} du'_{n-1} &= 0, \\ dv'_{n-2} &= u'_{n-1} + e_{n-1}(u'_{n-1}, u'_{n-1}), \\ dw'_{n+p-2} &= e_{n-p-1}(u'_{n-1}, u'_{n-1}), \\ dt'_{n+p-3} &= w'_{n+p-2} + e_{n+p-2}(w'_{n+p-2}, w'_{n+p-2}) \\ &\quad + e_{n+p-1}(e_{n-p-1}(u'_{n-1}, u'_{n-1}), w'_{n+p-2}) + \gamma'_{n+p-2}. \end{aligned}$$

Note that we have recovered algebraically the fact that ΩX_1 can be obtained as the pull-back

$$\begin{array}{ccc} \Omega X_1 & \longrightarrow & PK(\mathbf{Z}/2, n+p-1) \\ \downarrow & & \downarrow \\ K(\mathbf{Z}/2, n-1) & \longrightarrow & K(\mathbf{Z}/2, n+p-1) \end{array}$$

where the bottom horizontal map is also Sq^p.

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